Int J Theor Phys (2009) 48: 689–692
DOI 10.1007/s10773-008-9844-5

1-Soliton Solution of 1+2 Dimensional Nonlinear Schrödinger's Equation in Kerr Law Media

Anjan Biswas

Received: 18 June 2008 / Accepted: 29 August 2008 / Published online: 9 September 2008

Abstract This paper integrates the nonlinear Schrödinger's equation in 1+2 dimensions with Kerr law nonlinearity. An exact 1-soliton solution is obtained in closed form using the solitary wave ansatz. Finally, the consertved quantities are calculated using this soliton solution.

Keywords Optical solitons · Integrability · Conserved quantities · Kerr law

1 Introduction

The research on optical solitons has been going on for the past few decades and there has been a lot of progress in this direction [1–10]. The main governing equation in this area of study is the nonlinear Schrödinger's equation (NLSE) [1, 2, 4, 10]. This equation has mostly been studied in 1+1 dimensions as it is integrable by the classical method of Inverse Scattering Transform (IST) [1]. NLSE is a nonlinear evolution equation that has been integrated by the method of IST. Although this method is a powerful technique of integrating nonlinear evolutions equations, it can be used to integrate very few such equations. Thus, necessity had lead to the invention of many modern methods of integrating these equations. Thanks to these modern methods of integrability and gone are those days of IST monopoly!

There are various advantages and dis-advantages of these modern methods of integrability that includes Wadati trace method, tanh-coth method, variational iteration method, exponential function method and many more. Although a closed form soliton solution can be obtained by these techniques, the disadvantage of these methods is that these techniques cannot compute the conserved quantities of nonlinear evolution equations nor it can lay down an expression of the soliton radiation. Nevertheless, the fact that soliton solutions can be obtained is itself a big blessing. In this paper, the NLSE in 1+2 dimensions, with Kerr law nonlinearity, will be integrated using a similar such modern technique that otherwise fails the Painleve test of integrability and therefore does not belong to the IST picture.

A. Biswas (✉)
Center for Research and Education in Optical Sciences and Applications, Department of Applied Mathematics and Theoretical Physics, Delaware State University, Dover, DE 19901-2277, USA
e-mail: biswas.anjan@gmail.com

2 Mathematical Analysis

The dimensionless form of the NLSE in 1+2 dimensions, with Kerr law nonlinearity, is given by [1]

$$iq_t + \frac{1}{2}\left(q_{xx} + q_{yy}\right) + |q|^2 q = 0 \tag{1}$$

Here in (1), the first term represents the evolution term, the second and third terms, in parenthesis, represent the dispersion in x and y directions while the third term represents nonlinearity. Solitons are the result of a delicate balance between dispersion and nonlinearity. In this paper, (1) will be integrated by the usage of solitary wave ansatze.

Thus, based on the soliton solution in 1+1 dimensions, the 1-soliton solution of (1) is taken to be given by [1, 4]

$$q(x, y, t) = \frac{A}{\cosh^p[(B_1 x + B_2 y - vt)]} e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)} \tag{2}$$

Here in (2), A is the amplitude of the soliton, B_1 is the inverse width in the x-direction and B_2 is the inverse width in the y-direction and v represents the velocity of the soliton. Also κ_1 and κ_2 represents the soliton frequency in the x and y directions respectively, while ω represents the solitary wave number and finally θ is the phase constant of the soliton. The exponent p, which is unknown at this point, will be determined in course of finding the exact soliton solution.

Now, from (2),

$$q_t = \left[pvA \frac{\tanh \tau}{\cosh^p \tau} + \frac{i\omega A}{\cosh^p \tau} \right] e^{i\phi} \tag{3}$$

$$q_{xx} = \left[\frac{p^2 A B_1^2}{\cosh^p \tau} - \frac{p(p+1) A B_1^2}{\cosh^{p+2} \tau} - \frac{\kappa_1^2 A}{\cosh^p \tau} + 2i\kappa_1 p A B_1 \frac{\tanh \tau}{\cosh^p \tau} \right] e^{i\phi} \tag{4}$$

and

$$q_{yy} = \left[\frac{p^2 A B_2^2}{\cosh^p \tau} - \frac{p(p+1) A B_2^2}{\cosh^{p+2} \tau} - \frac{\kappa_2^2 A}{\cosh^p \tau} + 2i\kappa_2 p A B_2 \frac{\tanh \tau}{\cosh^p \tau} \right] e^{i\phi} \tag{5}$$

where $\tau = B_1 x + B_2 y - vt$. Substituting (3)–(5) into (1), and equating the real and imaginary parts, yields the following pair of relations

$$\kappa_1 p A B_1 + \kappa_2 p A B_2 + pvA = 0 \tag{6}$$

$$\begin{aligned} -\frac{\omega A}{\cosh^p \tau} + \frac{p^2 A B_1^2}{2\cosh^p \tau} - \frac{p(p+1) A B_1^2}{2\cosh^{p+2} \tau} - \frac{\kappa_1^2 A}{2\cosh^p \tau} \\ + \frac{p^2 A B_2^2}{2\cosh^p \tau} - \frac{p(p+1) A B_2^2}{2\cosh^{p+2} \tau} - \frac{\kappa_2^2 A}{2\cosh^p \tau} + \frac{A^3}{\cosh^{3p} \tau} = 0 \end{aligned} \tag{7}$$

From (6), it can be seen that

$$v = -\left(\kappa_1 B_1 + \kappa_2 B_2\right) \tag{8}$$

Now, from (7) equating the exponents $3p$ and $p + 2$ yields

$$p = 1 \tag{9}$$

Again, equating the coefficients of $\cosh^{p+2}\tau$ from the third, sixth and eighth terms in (7), gives

$$A = \sqrt{B_1^2 + B_2^2} \tag{10}$$

Finally, equating the coefficients of $\cosh^{p}\tau$ from the first, second, fourth, fifth and seventh terms in (7), yield

$$\omega = \frac{1}{2}\left[\left(B_1^2 + B_2^2\right) - \left(\kappa_1^2 + \kappa_2^2\right)\right] \tag{11}$$

Thus, the 1-soliton solution of the NLSE in 1+2 dimensions with Kerr law nonlinearity is given by

$$q(x, y, t) = \frac{A}{\cosh[(B_1 x + B_2 y - vt)]} e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)} \tag{12}$$

where the amplitude is related to the widths in the x- and y-directions as given by (10), the velocity of the soliton is given in (8) and finally the wave number is given by (11).

3 Integrals of Motion

The NLSE in 1+2 dimensions, with Kerr law nonlinearity has at least 3 integrals of motion. They are the power (P), linear momentum (M) and the Hamiltonian (H). These are respectively given by [1]

$$P = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |q|^2\, dx dy = \frac{2A^2}{B_1 B_2} \tag{13}$$

$$M = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\left(qq_x^* - q^* q_x\right) + \left(qq_y^* - q^* q_y\right)\right\} dx dy = \frac{2(\kappa_1 + \kappa_2) A^2}{B_1 B_2} \tag{14}$$

$$\begin{aligned} H &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(|q_x|^2 + |q_y|^2 - |q|^4\right) dx dy \\ &= \frac{A^2}{2B_1 B_2}\left[2\left(B_1^2 + B_2^2 + 2\kappa^2\right) - A^2\left(B_1^2 + B_2^2 - 1\right)\right] \end{aligned} \tag{15}$$

In order to calculate these conserved quantities, the 1-soliton solution given by (12) is used.

4 Conclusions

This paper obtains the 1-soliton solution of the NLSE in 1+2 dimensions with Kerr law nonlinearity. The solitary wave ansatze is used to carry out the integration of this equation as it was pointed out the classical method of IST will not work in this case. In future, the perturbations terms of this equation will be considered and will be studied using the soliton perturbation theory. In addition to the deterministic perturbation terms, stochastic perturbation terms will be considered too.

Acknowledgements This research was fully supported by NSF-CREST Grant No: HRD-0630388 and the support is genuinely and sincerely appreciated.

References

1. Ablowitz, M.J., Segur, H.: Solitons and the Inverse Scattering Transform. SIAM, Philadelphia (1981)
2. Akhmediev, N.N., Ankiewicz, A., Grimshaw, R.: Hamiltonian-versus-energy diagrams in soliton theory. Phys. Rev. E **59**(5), 6088–6096 (1999)
3. Asif, N., Shwetanshumala, S., Konar, S.: Photovoltaic spatial soliton pairs in two-photon photorefractive materials. Phys. Lett. A **372**(5), 735–740 (2008)
4. Biswas, A., Aceves, A.: Dynamics of solitons in optical fibers. J. Mod Opt. **48**(7), 1135–1150 (2001)
5. Fewo, S.I., Kofane, T.C.: A collective variable approach for optical solitons in the cubic-quintic complex Ginzburg-Landau equation with third order dispersion. Opt. Commun. **282**(10), 2893–2906 (2008)
6. Jana, S., Konar, S.: A new family of Thirring type optical spatial solitons via electromagnetically induced transparency. Phys. Lett. A **362**(5–6), 435–438 (2007)
7. Konar, S., Mishra, M., Jana, S.: Nonlinear evolution of cosh-Gaussian laser beams and generation of flat top spatial solitons in cubic quintic nonlinear media. Phys. Lett. A **362**(5–6), 505–510 (2007)
8. Shwetanshumala, S.: Temporal solitons of modified complex Ginzburg-Landau equation. Prog. Electromagn. Res. Lett. **3**, 17–24 (2008)
9. Yang, J., Kaup, D.J.: Stability and evolution of solitary waves in perturbed generalized nonlinear Schrödinger's equations. SIAM J. Appl. Math. **60**(3), 967–989 (2000)
10. Zhidkov, P.E.: Korteweg-de Vries and Nonlinear Schrödinger's Equations: Qualitative Theory. Springer, New York (2001)